HAMILTON-SOUPLET-ZHANG'S GRADIENT ESTIMATES FOR TWO TYPES OF NONLINEAR PARABOLIC EQUATIONS UNDER THE RICCI FLOW

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Abstract. In this paper, we consider gradient estimates for two types of nonlinear parabolic equations under the Ricci flow: one is the equation

$$u_t = \Delta u + au \log u + bu$$

with a, b two real constants, the other is

$$u_t = \Delta u + \lambda u^{\alpha}$$

with λ, α two real constants. By a suitable scaling for the above two equations, we obtain Hamilton-Souplet-Zhang type gradient estimates.

1. Introduction

Since the nonlinear parabolic equation

$$u_t = \Delta u + au \log u + bu \tag{1.1}$$

on a given Riemannian manifold is related to gradient Ricci solitons which are self-similar solutions to the Ricci flow, many attentions are paid to the study on gradient estimates for the equation (1.1), for example, see [5–7, 9, 10, 12, 17]. Here a, b in (1.1) are two real constants. Clearly, a heat equation

$$u_t = \Delta u \tag{1.2}$$

is a special case of (1.1) when $a = b = 0$. Hence many known results on heat equations are generalized to the nonlinear parabolic equation (1.1). For gradient estimates of solutions to (1.1) of Li-Yau type, Davies type, Hamilton type and Li-Xu type on a given Riemannian manifold, we refer to [2, 5, 7, 17] and the references therein. In a recent paper [3], Dung and Khanh obtained Hamilton-Souplet-Zhang type gradient estimates on a given Riemannian manifold for (1.1). On a family of Riemannian metrics $g(t)$ evolving by the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \tag{1.3}$$

Hsu in [8] obtained Li-Yau type gradient estimates of (1.1).

In [15], generalizing Hamilton's estimate in [4], Souplet and Zhang proved

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Theorem A [15]. *Let (M^n, g) be an n-dimensional Riemannian manifold with $\mathrm{Ric}(M^n) \geq -k$, where k is a non-negative constant. Suppose that u is a positive solution to the equation* (1.2) *in $Q_{R,T} = \{(x,t) \,|\, x \in M,\, d(x, x_0, t) < R,\, t \in [0,T]\}$ with $u \leq A$. Then in $Q_{R,T}$,*

$$\frac{|\nabla u|}{u} \leq C\left(\sqrt{k} + \frac{1}{R} + \frac{1}{\sqrt{T}}\right)\left(1 + \log\frac{A}{u}\right), \tag{1.4}$$

where the constant C depends only on the dimension n.

The key to prove Theorem A of Souplet and Zhang is the scaling $u \to \tilde{u} = u/A$. After this scaling, (1.2) becomes the following heat equation with respect to $\tilde{u}$:

$$\tilde{u}_t = \Delta \tilde{u} \tag{1.5}$$

since the heat equation is linear. Under this case, we obtain that $0 < \tilde{u} \leq 1$. Inspired by this method, in this paper, we also adopt the similar scaling method by $u \to \tilde{u} = u/A$ to study the nonlinear parabolic equation (1.1). By the scaling, we can derive from (1.1) the following analogous equation:

$$\tilde{u}_t = \Delta \tilde{u} + a\tilde{u}\log\tilde{u} + \tilde{b}\tilde{u}, \tag{1.6}$$

where the constant $\tilde{b}$ satisfies $\tilde{b} = b + a\log A$. That is, we only need to study the nonlinear equation (1.6) with $0 < \tilde{u} \leq 1$.

Our first result is the following Hamilton-Souplet-Zhang type gradient estimates of the nonlinear equation (1.1) under the Ricci flow:

Theorem 1.1. *Let M be a complete Riemannian manifold with a family of Riemannian metric $g(t)$ evolving by the Ricci flow* (1.3)*. Suppose that u is a positive solution to* (1.1) *in*

$$B_{R,T} = \{(x,t) \,|\, x \in M,\, d(x, x_0, t) < R,\, t \in [0,T]\} \tag{1.7}$$

with $|\mathrm{Ric}| \leq k$ for some positive constant k and $u \leq A$. Then there exists a constant C depending only on the dimension of M such that

$$\frac{|\nabla u|}{u} \leq C\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + \sqrt{M_{a,b}}\right)\left(1 + \log\frac{A}{u}\right) \tag{1.8}$$

for all $(x,t) \in B_{\frac{R}{2},T}$ with $t \neq 0$, where $M_{a,b} = \max\{0, a(1 + \log A) + b\}$.

The study to Li-Yau type estimates of the following nonlinear parabolic equation

$$u_t = \Delta u + \lambda u^{\alpha}, \tag{1.9}$$

where λ, α are two real constants, can be traced back to Li [13]. Later, for $0 < \alpha < 1$, Zhu in [21] obtained Hamilton-Souplet-Zhang type gradient estimates of (1.9) on a given Riemannian manifold. On gradient estimates of the elliptic case of (1.9), see [18,20]. A natural subject is to study Hamilton-Souplet-Zhang type gradient estimates of the nonlinear equation (1.9) under the Ricci flow. Our second result is the following:

Theorem 1.2. *Let M be a complete Riemannian manifold with a family of Riemannian metric $g(t)$ evolving by the Ricci flow* (1.3)*. Suppose that u is a positive solution to* (1.9) *in*

$$B_{R,T} = \{(x,t) \,|\, x \in M,\, d(x,x_0,t) < R,\, t \in [0,T]\} \tag{1.10}$$

with $|\mathrm{Ric}| \leq k$ for some positive constant k and $u \leq B$. Then there exists a constant C depending only on the dimension of M such that

1) if $\alpha \geq 1$, then

$$\frac{|\nabla u|}{u} \leq C\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + \sqrt{M_\lambda \alpha}\right)\left(1 + \log \frac{B}{u}\right), \tag{1.11}$$

for all $(x,t) \in B_{\frac{R}{2},\frac{T}{2}}$ with $t \neq 0$, where $M_\lambda = \max\{0, \lambda B^{\alpha-1}\}$;

2) if $\alpha \leq 0$, then

$$\frac{|\nabla u|}{u} \leq C\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + \sqrt{M_\lambda(-\alpha+1)u_{\min}^{\alpha-1}}\right)\left(1 + \log \frac{B}{u}\right), \tag{1.12}$$

for all $(x,t) \in B_{\frac{R}{2},T}$ with $t \neq 0$, where $M_\lambda = \max\{0, -\lambda\}$;

3) if $\alpha \in (0,1)$, then

$$\frac{|\nabla u|}{u} \leq C\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + \sqrt{|\lambda| u_{\min}^{\alpha-1}}\right)\left(1 + \log \frac{B}{u}\right) \tag{1.13}$$

for all $(x,t) \in B_{\frac{R}{2},T}$ with $t \neq 0$.

Remark 1.1. Taking $a = b = 0$ in (1.8), we obtain the estimate (2.3) of Theorem 2.2 in [1] with respect to the heat equation under the Ricci flow. Hence, our estimates in Theorem 1.1 extend Bailesteanu, Cao and Pulemotov's Theorem 2.2.

Remark 1.2. There are many studies on gradient estimates of the heat equation (1.2) under geometric flows, we refer to [14, 16] and among others.

2. Proof of Theorem 1.1

In order to prove our Theorem 1.1, we first give a lemma which will play an important role in the proof.

Lemma 2.1. *Let M be a complete Riemannian manifold with a family of Riemannian metric $g(t)$ evolving by the Ricci flow* (1.3)*. Let u be a positive solution to* (1.1) *with $u \leq A$. Denote by $\tilde{u} = u/A$, $f = \log \tilde{u} \leq 0$ and $w = |\nabla \log(1-f)|^2$. Then, it holds*

$$(\Delta - \partial_t)w \geq \frac{2f}{1-f}\nabla f \nabla w + 2(1-f)w^2 - \frac{2(a+B)}{1-f}w, \tag{2.1}$$

where $B = b + a\log A$.

Proof. Under the scaling $u \to \tilde{u} = u/A$, we have $0 < \tilde{u} \le 1$. From (1.1), we obtain that $\tilde{u}$ satisfies the following equation

$$\tilde{u}_t = \Delta\tilde{u} + a\tilde{u}\log\tilde{u} + B\tilde{u}, \tag{2.2}$$

where the constant B satisfies $B = b + a\log A$. Let $f = \log\tilde{u} \le 0$ and $w = |\nabla \log(1-f)|^2 = \frac{|\nabla f|^2}{(1-f)^2}$. Then we have

$$f_t = \Delta f + |\nabla f|^2 + af + B. \tag{2.3}$$

Using (1.3), we can obtain

$$(|\nabla f|^2)_t = 2R_{ij}f_if_j + 2f_i(f_t)_i. \tag{2.4}$$

By the definition of w, we have

$$\begin{aligned} w_t =& \frac{2}{(1-f)^2}[R_{ij}f_if_j + f_i(f_t)_i] + \frac{2}{(1-f)^3}f_j^2 f_t \\ =& \frac{2}{(1-f)^2}[R_{ij}f_if_j + f_if_{jji} + 2f_{ij}f_if_j + af_i^2] \\ &+ \frac{2}{(1-f)^3}f_j^2(f_{ii} + f_i^2 + af + B). \end{aligned} \tag{2.5}$$

On the other hand,

$$\begin{aligned} \Delta w =& \frac{6}{(1-f)^4}f_i^2f_j^2 + \frac{2}{(1-f)^3}f_{ii}f_j^2 + \frac{8}{(1-f)^3}f_{ji}f_if_j \\ &+ \frac{2}{(1-f)^2}f_{ji}^2 + \frac{2}{(1-f)^2}f_jf_{jii} \\ =& \frac{6}{(1-f)^4}f_i^2f_j^2 + \frac{2}{(1-f)^3}f_{ii}f_j^2 + \frac{8}{(1-f)^3}f_{ji}f_if_j \\ &+ \frac{2}{(1-f)^2}f_{ji}^2 + \frac{2}{(1-f)^2}f_jf_{iij} + \frac{2}{(1-f)^2}R_{ij}f_if_j, \end{aligned} \tag{2.6}$$

where, in the second equality, we used the Ricci formula:

$$f_{jii} = f_{iji} = f_{iij} + R_{ij}f_i.$$

By the formulas (2.5) and (2.6), we arrive at

$$\begin{aligned} (\Delta - \partial_t)w =& \frac{2}{(1-f)^2}f_{ji}^2 + \frac{8}{(1-f)^3}f_{ji}f_if_j - \frac{4}{(1-f)^2}f_{ij}f_if_j \\ &+ \frac{6}{(1-f)^4}f_i^2f_j^2 - \frac{2}{(1-f)^3}f_i^2f_j^2 - \frac{2(a+B)}{1-f}\frac{f_i^2}{(1-f)^2}. \end{aligned} \tag{2.7}$$

Note that

$$\nabla f \nabla w = \frac{2}{(1-f)^3}f_i^2f_j^2 + \frac{2}{(1-f)^2}f_{ji}f_if_j. \tag{2.8}$$

Therefore, (2.7) can be written as

$$
\begin{aligned}
(\Delta-\partial_t)w =& \frac{2}{(1-f)^2}f_{ji}^2+\frac{4}{(1-f)^3}f_{ji}f_if_j-\frac{4}{(1-f)^2}f_{ij}f_if_j\\
&+\frac{2}{1-f}\nabla f\nabla w+2\frac{|\nabla f|^4}{(1-f)^4}-2\frac{|\nabla f|^4}{(1-f)^3}-\frac{2(a+B)}{1-f}\frac{|\nabla f|^2}{(1-f)^2}\\
=& \frac{2}{(1-f)^2}f_{ji}^2+\frac{4}{(1-f)^3}f_{ji}f_if_j-2\nabla f\nabla w\\
&+\frac{2}{1-f}\nabla f\nabla w+2\frac{|\nabla f|^4}{(1-f)^4}+2\frac{|\nabla f|^4}{(1-f)^3}-\frac{2(a+B)}{1-f}\frac{|\nabla f|^2}{(1-f)^2}\\
=& \frac{2}{(1-f)^2}\left(f_{ji}+\frac{1}{1-f}f_if_j\right)^2\\
&+\frac{2f}{1-f}\nabla f\nabla w+2\frac{|\nabla f|^4}{(1-f)^3}-\frac{2(a+B)}{1-f}\frac{|\nabla f|^2}{(1-f)^2}\\
\geq& \frac{2f}{1-f}\nabla f\nabla w+2(1-f)w^2-\frac{2(a+B)}{1-f}w.
\end{aligned}
$$

Then, the desired estimate (2.1) follows. □

Proof of Theorem 1.1. Let $\psi=\psi(x,t)$ be a smooth cutoff function supported in $B_{R,T}$ which satisfies the following properties (see Page 13 in [19], or Lemma 2.1 in [1]):

1) $\psi=\psi(d(x,x_0,t),t)\equiv\psi(r,t)$; $\psi(x,t)=1$ in $B_{\frac{R}{2},\frac{T}{2}}$ and $0\leq\psi\leq 1$.
2) ψ is decreasing as a radial function in the spatial variables.
3) $|\frac{\partial\psi}{\partial r}|\leq\frac{C_\gamma}{R}\psi^\gamma$ and $|\frac{\partial^2\psi}{\partial r^2}|\leq\frac{C_\gamma}{R^2}\psi^\gamma$ for every $\gamma\in(0,1)$.
4) $|\frac{\partial\psi}{\partial t}|\leq\frac{C}{T}\psi^{\frac{1}{2}}$.

We use (2.1) to conclude that

$$
\begin{aligned}
(\Delta-\partial_t)(\psi w) =& \psi(\Delta-\partial_t)w+w(\Delta-\partial_t)\psi+2\nabla\psi\nabla w\\
=& \psi(\Delta-\partial_t)w+w(\Delta-\partial_t)\psi\\
&+2\frac{\nabla\psi}{\psi}\nabla(\psi w)-2w\frac{|\nabla\psi|^2}{\psi}\\
\geq& \frac{2f}{1-f}[\nabla f\nabla(\psi w)-w\nabla f\nabla\psi]+2(1-f)\psi w^2 \qquad (2.9)\\
&-\frac{2(a+B)}{1-f}\psi w+w(\Delta-\partial_t)\psi\\
&+2\frac{\nabla\psi}{\psi}\nabla(\psi w)-2w\frac{|\nabla\psi|^2}{\psi}.
\end{aligned}
$$

Now we let (x_1,t_1) be a maximum point of ψw in the closure of $B_{R,T}$, and where $\psi w>0$ (otherwise the proof is trivial). Then at the point (x_1,t_1), we

have $\Delta(\psi w) \leq 0$, $(\psi w)_t \geq 0$ and $\nabla(\psi w) = 0$. Thus, from (2.9), we deduce

$$\begin{aligned} 2(1-f)\psi w^2 \leq & \frac{2f}{1-f} w \nabla f \nabla \psi + \frac{2(a+B)}{1-f}\psi w \\ & - w(\Delta - \partial_t)\psi + 2w\frac{|\nabla\psi|^2}{\psi} \end{aligned} \tag{2.10}$$

at (x_1, t_1). Next, we will find an upper bound for each term of the right hand side of (2.10). By the Cauchy inequality, we have

$$\begin{aligned} \frac{2f}{1-f} w \nabla f \nabla \psi \leq & 2|f| w^{\frac{3}{2}} |\nabla\psi| \\ = & 2[\psi(1-f)w^2]^{\frac{3}{4}} \frac{f|\nabla\psi|}{[\psi(1-f)]^{\frac{3}{4}}} \\ \leq & (1-f)\psi w^2 + C\frac{1}{R^4}\frac{f^4}{(1-f)^3}. \end{aligned} \tag{2.11}$$

It has been shown in [19](see formulas (3.30), (3.32) and (3.34) in [19]) that:

$$w\frac{|\nabla\psi|^2}{\psi} \leq \frac{1}{8}\psi w^2 + C\frac{1}{R^4}, \tag{2.12}$$

$$-(\Delta\psi)w \leq \frac{1}{8}\psi w^2 + C\frac{1}{R^4} + C\frac{k}{R^2} \tag{2.13}$$

and

$$\left|\frac{\partial\psi}{\partial t}\right| w \leq \frac{1}{8}\psi w^2 + C\frac{1}{T^2} + Ck^2. \tag{2.14}$$

Putting (2.11)-(2.14) into (2.10), we obtain

$$\begin{aligned} (1-f)\psi w^2 \leq & C\frac{1}{R^4}\frac{f^4}{(1-f)^3} + \frac{2(a+B)}{1-f}\psi w \\ & + \frac{1}{2}\psi w^2 + C\frac{1}{R^4} + C\frac{1}{T^2} + Ck^2. \end{aligned} \tag{2.15}$$

Since $f \leq 0$, the inequality (2.15) implies

$$\begin{aligned} \psi w^2 \leq & C\left(\frac{1}{R^4}\frac{f^4}{(1-f)^4} + \frac{1}{R^4} + \frac{1}{T^2} + k^2 + (\max\{0, a+B\})^2\right) \\ = & C\left(\frac{1}{R^4}\frac{f^4}{(1-f)^4} + \frac{1}{R^4} + \frac{1}{T^2} + k^2 + M_{a,b}^2\right). \end{aligned} \tag{2.16}$$

Therefore, we deduce that, for any $(x,t) \in B_{R,T}$,

$$\begin{aligned} (\psi^2 w^2)(x,t) \leq & (\psi w^2)(x_1, t_1) \\ \leq & C\left(\frac{1}{R^4} + \frac{1}{T^2} + k^2 + M_{a,b}^2\right) \end{aligned} \tag{2.17}$$

from $\frac{f^4}{(1-f)^4} < 1$. Noticing $\psi(x,t) = 1$ in $B_{\frac{R}{2},\frac{T}{2}}$ and $w = \frac{|\nabla f|^2}{(1-f)^2}$, we have

$$\frac{|\nabla f|}{1-f} \le C\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + \sqrt{M_{a,b}}\right) \tag{2.18}$$

holds at any $(x,t) \in B_{\frac{R}{2},T}$, which shows that

$$\frac{|\nabla u|}{u} \le C\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + \sqrt{M_{a,b}}\right)\left(1 + \log\frac{A}{u}\right). \tag{2.19}$$

We complete the proof of the estimate (1.8) in Theorem 1.1.

3. Proof of Theorem 1.2

As in the proof of Theorem 1.1, we first give a key lemma:

Lemma 3.1. *Let M be a complete Riemannian manifold with a family of Riemannian metric $g(t)$ evolving by the Ricci flow* (1.3)*. Let u be a positive solution to* (1.9) *with $u \le B$. Denote by $\tilde{u} = u/B$, $f = \log\tilde{u} \le 0$ and $w = |\nabla\log(1-f)|^2$. Then, it holds*

$$(\Delta - \partial_t)w \ge \frac{2f}{1-f}\langle\nabla f, \nabla w\rangle + 2(1-f)w^2 - 2\tilde{\lambda}\Big(\alpha - \frac{-f}{1-f}\Big)e^{(\alpha-1)f}w, \tag{3.1}$$

where $\tilde{\lambda} = \lambda B^{\alpha-1}$.

Proof. By the scaling $u \to \tilde{u} = u/B$, we have $0 < \tilde{u} \le 1$. Therefore, we obtain from (1.9) that $\tilde{u}$ satisfies

$$\tilde{u}_t = \Delta\tilde{u} + \tilde{\lambda}\tilde{u}^{\alpha}, \tag{3.2}$$

where $\tilde{\lambda} = \lambda B^{\alpha-1}$. Let $f = \log\tilde{u} \le 0$ and

$$w = |\nabla\log(1-f)|^2. \tag{3.3}$$

Then, the function f satisfies

$$f_t = \Delta f + |\nabla f|^2 + \tilde{\lambda}e^{(\alpha-1)f}. \tag{3.4}$$

Using (1.3) again, one has

$$(|\nabla f|^2)_t = 2R_{ij}f_if_j + 2f_i(f_t)_i. \tag{3.5}$$

It follows from (3.3) that

$$\begin{aligned} w_t =& \frac{2}{(1-f)^2}[R_{ij}f_if_j + f_i(f_t)_i] + \frac{2}{(1-f)^3}f_j^2f_t \\ =& \frac{2}{(1-f)^2}[R_{ij}f_if_j + f_if_{jji} + 2f_{ij}f_if_j + \tilde{\lambda}(\alpha-1)e^{(\alpha-1)f}f_i^2] \\ &+ \frac{2}{(1-f)^3}f_j^2(f_{ii} + f_i^2 + \tilde{\lambda}e^{(\alpha-1)f}). \end{aligned} \tag{3.6}$$

Similarly, by the Ricci formula, we obtain

$$\begin{aligned}\Delta w =&\frac{6}{(1-f)^4}f_i^2f_j^2+\frac{2}{(1-f)^3}f_{ii}f_j^2+\frac{8}{(1-f)^3}f_{ji}f_if_j\\&+\frac{2}{(1-f)^2}f_{ji}^2+\frac{2}{(1-f)^2}f_jf_{iij}+\frac{2}{(1-f)^2}R_{ij}f_if_j.\end{aligned}\tag{3.7}$$

Thus, we derive from (3.6) and (3.7)

$$\begin{aligned}(\Delta-\partial_t)w =&\frac{2}{(1-f)^2}f_{ji}^2+\Big[\frac{6}{(1-f)^4}-\frac{2}{(1-f)^3}\Big]f_i^2f_j^2\\&+\Big[\frac{8}{(1-f)^3}-\frac{4}{(1-f)^2}\Big]f_{ji}f_if_j\\&-2\tilde{\lambda}\Big[(\alpha-1)\frac{1}{(1-f)^2}+\frac{1}{(1-f)^3}\Big]e^{(\alpha-1)f}f_i^2.\end{aligned}\tag{3.8}$$

Using the relationship

$$\langle\nabla f,\nabla w\rangle=\frac{2}{(1-f)^3}f_i^2f_j^2+\frac{2}{(1-f)^2}f_{ji}f_if_j,\tag{3.9}$$

(3.8) can be written as

$$\begin{aligned}&(\Delta-\partial_t)w-\varepsilon\langle\nabla f,\nabla w\rangle\\&=\frac{2}{(1-f)^2}f_{ji}^2+2\Big[\frac{3}{(1-f)^4}-(1+\varepsilon)\frac{1}{(1-f)^3}\Big]f_i^2f_j^2\\&\quad+2\Big[\frac{4}{(1-f)^3}-(2+\varepsilon)\frac{1}{(1-f)^2}\Big]f_{ji}f_if_j\\&\quad-2\tilde{\lambda}\Big[(\alpha-1)\frac{1}{(1-f)^2}+\frac{1}{(1-f)^3}\Big]e^{(\alpha-1)f}f_i^2,\end{aligned}\tag{3.10}$$

where $\varepsilon=\varepsilon(f)$ is a function depending on f which will be determined. Applying the inequality

$$\begin{aligned}&\frac{2}{(1-f)^2}f_{ji}^2+2\Big[\frac{4}{(1-f)^3}-(2+\varepsilon)\frac{1}{(1-f)^2}\Big]f_{ji}f_if_j\\&\quad=\frac{2}{(1-f)^2}\Big\{f_{ji}^2+\Big[\frac{4}{1-f}-(2+\varepsilon)\Big]f_{ji}f_if_j\Big\}\\&\quad\geq-\frac{1}{2(1-f)^2}\Big[\frac{4}{1-f}-(2+\varepsilon)\Big]^2f_i^2f_j^2,\end{aligned}$$

into (3.10) gives

$$\begin{aligned}
&(\Delta-\partial_t)w-\varepsilon\langle\nabla f,\nabla w\rangle\\
&\quad\geq\Big[-\frac{2}{(1-f)^4}+(6+2\varepsilon)\frac{1}{(1-f)^3}-(2+\varepsilon)^2\frac{1}{2(1-f)^2}\Big]f_i^2f_j^2\\
&\quad\quad-2\tilde{\lambda}\Big[(\alpha-1)\frac{1}{(1-f)^2}+\frac{1}{(1-f)^3}\Big]e^{(\alpha-1)f}f_i^2\\
&\quad=\frac{1}{(1-f)^4}\Big\{-\frac{1}{2}(1-f)^2\varepsilon^2-2[(1-f)^2-(1-f)]\varepsilon\\
&\quad\quad-[2(1-f)^2-6(1-f)+2]\Big\}f_i^2f_j^2\\
&\quad\quad-2\tilde{\lambda}\Big[(\alpha-1)\frac{1}{(1-f)^2}+\frac{1}{(1-f)^3}\Big]e^{(\alpha-1)f}f_i^2.
\end{aligned} \tag{3.11}$$

Taking

$$\varepsilon=-2+\frac{2}{1-f}$$

in (3.11), we derive

$$\begin{aligned}
&(\Delta-\partial_t)w-\frac{2f}{1-f}\langle\nabla f,\nabla w\rangle\\
&\quad\geq\frac{2}{(1-f)^3}|\nabla f|^4-2\tilde{\lambda}\Big[(\alpha-1)\frac{1}{(1-f)^2}+\frac{1}{(1-f)^3}\Big]e^{(\alpha-1)f}f_i^2\\
&\quad=2(1-f)w^2-2\tilde{\lambda}\Big(\alpha-\frac{-f}{1-f}\Big)e^{(\alpha-1)f}w.
\end{aligned} \tag{3.12}$$

Thus, the desired estimate (3.1) is attained. □

Proof of Theorem 1.2. Let $\psi=\psi(x,t)$ be a smooth cutoff function supported in $B_{R,T}$ which is defined in section 2. We use (3.1) to conclude that

$$\begin{aligned}
(\Delta-\partial_t)(\psi w)=&\psi(\Delta-\partial_t)w+w(\Delta-\partial_t)\psi\\
&+2\frac{\nabla\psi}{\psi}\nabla(\psi w)-2w\frac{|\nabla\psi|^2}{\psi}\\
\geq&-2\frac{-f}{1-f}[\nabla f\nabla(\psi w)-w\nabla f\nabla\psi]+2(1-f)\psi w^2\\
&-2\tilde{\lambda}\Big(\alpha-\frac{-f}{1-f}\Big)e^{(\alpha-1)f}\psi w+w(\Delta-\partial_t)\psi\\
&+2\frac{\nabla\psi}{\psi}\nabla(\psi w)-2w\frac{|\nabla\psi|^2}{\psi}.
\end{aligned} \tag{3.13}$$

Now we let (x_2,t_2) be a maximum point of ψw in the closure of $B_{R,T}$, and where $\psi w>0$ (otherwise the proof is trivial). Then at the point (x_2,t_2), we

have $\Delta(\psi w)\le 0$, $(\psi w)_t\ge 0$ and $\nabla(\psi w)=0$. Thus, from (3.13), we deduce

$$\begin{aligned}2(1-f)\psi w^2\le&\frac{2f}{1-f}w\nabla f\nabla\psi+2\tilde{\lambda}\Big(\alpha-\frac{-f}{1-f}\Big)e^{(\alpha-1)f}\psi w\\&-w(\Delta-\partial_t)\psi+2w\frac{|\nabla\psi|^2}{\psi}\end{aligned}\tag{3.14}$$

at (x_2,t_2).

Case one: $\alpha\ge 1$.

Note that

$$\frac{-f}{1-f}=1-\frac{1}{1-f}\in(0,1).\tag{3.15}$$

In this case, we have $\alpha-\frac{-f}{1-f}>0$ and $e^{(\alpha-1)f}\in(0,1)$. Hence

$$2\tilde{\lambda}\Big(\alpha-\frac{-f}{1-f}\Big)e^{(\alpha-1)f}\psi w\le C\frac{M_\lambda^2}{1-f}\Big(\alpha-\frac{-f}{1-f}\Big)^2+\frac{1}{16}(1-f)\psi w^2,\tag{3.16}$$

where $M_\lambda=\max\{0,\tilde{\lambda}\}$. Putting (3.16), (2.11)-(2.14) into (3.14), we obtain

$$\begin{aligned}(1-f)\psi w^2\le&C\Bigg[\frac{1}{R^4}\frac{f^4}{(1-f)^3}+\frac{M_\lambda^2}{1-f}\Big(\alpha-\frac{-f}{1-f}\Big)^2\\&+\frac{1}{R^4}+\frac{1}{T^2}+k^2\Bigg].\end{aligned}\tag{3.17}$$

Therefore, we deduce, for any $(x,t)\in B_{R,T}$,

$$\begin{aligned}(\psi^2w^2)(x,t)\le&(\psi w^2)(x_1,t_1)\\\le&C\left(\frac{1}{R^4}+\frac{1}{T^2}+k^2+M_\lambda^2\alpha^2\right)\end{aligned}\tag{3.18}$$

from $\frac{f^2}{(1-f)^2}<1$. Noticing $\psi(x,t)=1$ in $B_{\frac{R}{2},\frac{T}{2}}$ and $w=\frac{|\nabla f|^2}{(1-f)^2}$, we have

$$\frac{|\nabla f|}{1-f}\le C\left(\frac{1}{R}+\frac{1}{\sqrt{T}}+\sqrt{k}+\sqrt{M_\lambda\alpha}\right)\tag{3.19}$$

holds at any $(x,t)\in B_{\frac{R}{2},T}$, which shows that

$$\frac{|\nabla u|}{u}\le C\left(\frac{1}{R}+\frac{1}{\sqrt{T}}+\sqrt{k}+\sqrt{M_\lambda\alpha}\right)\left(1+\log\frac{B}{u}\right).\tag{3.20}$$

Case two: $\alpha\le 0$.

In this case, we have $\alpha-\frac{-f}{1-f}<0$ and $e^{(\alpha-1)f}>1$. Hence

$$2\tilde{\lambda}\Big(\alpha-\frac{-f}{1-f}\Big)e^{(\alpha-1)f}\psi w\le C\frac{\tilde{M}_\lambda^2}{1-f}\Big(\alpha-\frac{-f}{1-f}\Big)^2\tilde{u}_{\min}^{2(\alpha-1)}+\frac{1}{16}(1-f)\psi w^2,\tag{3.21}$$

where $\tilde{M}_\lambda = \max\{0, -\tilde{\lambda}\}$. Similarly, putting (3.21), (2.11)-(2.14) into (3.14), we obtain

$$\begin{aligned}(1-f)\psi w^2 \leq & C\Bigg[\frac{1}{R^4}\frac{f^4}{(1-f)^3} + \frac{\tilde{M}_\lambda^2}{1-f}\Big(\alpha - \frac{-f}{1-f}\Big)^2 \tilde{u}_{\min}^{2(\alpha-1)} \\ & + \frac{1}{R^4} + \frac{1}{T^2} + k^2\Bigg].\end{aligned} \tag{3.22}$$

Thus, we can obtain

$$\frac{|\nabla f|}{1-f} \leq C\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + \sqrt{\tilde{M}_\lambda(-\alpha+1)\tilde{u}_{\min}^{\alpha-1}}\right) \tag{3.23}$$

holds at any $(x,t) \in B_{\frac{R}{2},T}$, which shows that

$$\frac{|\nabla u|}{u} \leq C\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + \sqrt{M_\lambda(-\alpha+1)u_{\min}^{\alpha-1}}\right)\left(1 + \log\frac{B}{u}\right). \tag{3.24}$$

Case three: $\alpha \in (0,1)$.

In this case, we also have $e^{(\alpha-1)f} > 1$. Hence

$$2\tilde{\lambda}\Big(\alpha - \frac{-f}{1-f}\Big)e^{(\alpha-1)f}\psi w \leq C\frac{\tilde{\lambda}^2}{1-f}\Big(\alpha - \frac{-f}{1-f}\Big)^2 \tilde{u}_{\min}^{2(\alpha-1)} + \frac{1}{16}(1-f)\psi w^2. \tag{3.25}$$

Putting (3.25), (2.11)-(2.14) into (3.14), we obtain

$$\begin{aligned}(1-f)\psi w^2 \leq & C\Bigg[\frac{1}{R^4}\frac{f^4}{(1-f)^3} + \frac{\tilde{\lambda}^2}{1-f}\Big(\alpha - \frac{-f}{1-f}\Big)^2 \tilde{u}_{\min}^{2(\alpha-1)} \\ & + \frac{1}{R^4} + \frac{1}{T^2} + k^2\Bigg].\end{aligned} \tag{3.26}$$

Because of $\left(\alpha - \frac{-f}{1-f}\right)^2 \leq 1$, we can obtain

$$\frac{|\nabla f|}{1-f} \leq C\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + \sqrt{|\tilde{\lambda}|\tilde{u}_{\min}^{\alpha-1}}\right) \tag{3.27}$$

holds at any $(x,t) \in B_{\frac{R}{2},T}$, which shows that

$$\frac{|\nabla u|}{u} \leq C\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + \sqrt{|\lambda|u_{\min}^{\alpha-1}}\right)\left(1 + \log\frac{B}{u}\right). \tag{3.28}$$

We complete the proof of Theorem 1.2.

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